

AN UPPER DIAMETER BOUND FOR COMPACT RICCI SOLITONS WITH APPLICATIONS TO THE HITCHIN-THORPE INEQUALITY

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ABSTRACT. In this article, stimulated by Fernández-López and García-Río, we shall give an upper diameter bound for compact Ricci solitons in terms of the range of the scalar curvature. As an application, we shall provide some sufficient conditions for four-dimensional compact Ricci solitons to satisfy the Hitchin-Thorpe inequality.

1. INTRODUCTION

A *Ricci soliton* [9] is a complete Riemannian manifold (M, g) admitting a smooth vector field $X \in \mathfrak{X}(M)$ such that

$$\mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_X g = \lambda g \quad (1.1)$$

for some real constant $\lambda \in \mathbb{R}$, where Ric_g denotes the Ricci tensor of (M, g) and \mathcal{L}_X is the Lie derivative in the direction of X . The soliton (M, g) is said to be *shrinking*, *steady* and *expanding* if $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$, respectively. Typical examples of the Ricci soliton are Einstein manifolds, where X is given by a Killing vector field. In this case, we say that the soliton is *trivial*. Ricci solitons play an important role in the Ricci flow as they correspond to self-similar solutions and often arise as singularity models [3]. When X may be replaced by the gradient ∇f for some smooth function $f : M \rightarrow \mathbb{R}$, called a *potential function*, (M, g) is called a *gradient Ricci soliton*. In such a case, (1.1) becomes

$$\mathrm{Ric}_g + H_f = \lambda g, \quad (1.2)$$

where H_f denotes the Hessian of the function f . Due to Perelman [15], any compact Ricci soliton is a gradient one. It is well-known [3] that compact steady and expanding Ricci solitons must be trivial, as well as compact shrinking Ricci solitons in dimension two and three [3]. Examples of non-trivial compact Kähler-Ricci solitons were constructed by Koiso [10], Cao [2] and Wang and Zhu [16].

A lower diameter bound for compact shrinking Ricci solitons has been recently investigated by many authors [1, 4, 5, 7, 8]. In particular, a universal lower bound for compact shrinking Ricci solitons was first given by Futaki and Sano [8] in relation to study of the first non-zero eigenvalue of the Witten-Laplacian. On the other hand, Fernández-López and García-Río [5] gave the following lower diameter bound in terms of the Ricci curvature and the range of the potential function.

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Theorem 1.3 (Fernández-López and García-Río [5]). *Let (M, g) be an n -dimensional compact connected shrinking Ricci soliton satisfying (1.2). Then*

$$\text{diam}(M, g) \geq \max \left\{ \sqrt{\frac{2(f_{\max} - f_{\min})}{C - \lambda}}, \sqrt{\frac{2(f_{\max} - f_{\min})}{\lambda - c}}, 2\sqrt{\frac{2(f_{\max} - f_{\min})}{C - c}} \right\},$$

where f_{\max} and f_{\min} respectively denote the maximum and the minimum value of the potential function on the soliton.

In Theorem 1.3, the number

$$C := \max_{v \in TM} \{\text{Ric}_g(v, v) : |v| = 1\} \quad \text{and} \quad c := \min_{v \in TM} \{\text{Ric}_g(v, v) : |v| = 1\}$$

respectively denote the maximum and the minimum value of the Ricci curvature on the unit sphere bundle of (M, g) . Note that $cg \leq \text{Ric}_g \leq Cg$.

When the soliton has positive Ricci curvature, this diameter bound can be written in terms of the range of the scalar curvature as follows.

Corollary 1.4 (Fernández-López and García-Río [5]). *Let (M, g) be an n -dimensional compact connected shrinking Ricci soliton with positive Ricci curvature satisfying (1.2). Then*

$$\text{diam}(M, g) \geq \max \left\{ \sqrt{\frac{R_{\max} - R_{\min}}{\lambda(C - \lambda)}}, \sqrt{\frac{R_{\max} - R_{\min}}{\lambda(\lambda - c)}}, 2\sqrt{\frac{R_{\max} - R_{\min}}{\lambda(C - c)}} \right\},$$

where R_{\max} and R_{\min} respectively denote the maximum and the minimum value of the scalar curvature on the soliton.

Moreover, stimulated by the Myers diameter estimate [17, Theorem 1.4] via Bakry-Émery Ricci curvature, Fernández-López and García-Río [5] mentioned that an upper diameter bound for compact shrinking Ricci solitons would be given in terms of the range of the potential function, as well as in terms of the range of the scalar curvature.

The aim of this article is to give a positive answer to this conjecture by giving the following.

Theorem 1.5. *Let (M, g) be an n -dimensional compact connected shrinking Ricci soliton satisfying (1.2). Then*

$$\text{diam}(M, g) \leq \frac{1}{\lambda} \left(2\sqrt{R_{\max} - R_{\min}} + \sqrt{4(R_{\max} - R_{\min}) + (n - 1)\lambda\pi^2} \right). \quad (1.6)$$

When the soliton has positive Ricci curvature, this diameter bound can be written in terms of the range of the potential function as follows.

Corollary 1.7. *Let (M, g) be an n -dimensional compact connected shrinking Ricci soliton with positive Ricci curvature satisfying (1.2). Then*

$$\text{diam}(M, g) \leq 2\sqrt{\frac{2(f_{\max} - f_{\min})}{\lambda}} + \sqrt{\frac{8(f_{\max} - f_{\min}) + (n - 1)\pi^2}{\lambda}}.$$

Just as in the Einstein manifold, we may expect some topological obstruction to the existence of compact Ricci solitons. A validity of the Hitchin-Thorpe inequality for four-dimensional compact shrinking Ricci solitons was first shown by Ma [13] assuming some upper bounds on the L^2 -norm of the scalar curvature. On the other hand, Fernández-López and García-Río [6] investigated the same validity assuming the following upper diameter bounds in terms of the Ricci curvature.

Theorem 1.8 (Fernández-López and García-Río [5]). *Let (M, g) be a four-dimensional compact connected shrinking Ricci soliton satisfying (1.2). If*

$$\text{diam}(M, g) \leq \max \left\{ \sqrt{\frac{2}{C - \lambda}}, \sqrt{\frac{2}{\lambda - c}}, 2\sqrt{\frac{2}{C - c}} \right\},$$

then the soliton satisfies the Hitchin-Thorpe inequality $2\chi(M) \geq 3|\tau(M)|$.

The following corollary of Theorem 1.5 provides a sufficient condition for four-dimensional compact shrinking Ricci solitons to satisfy the Hitchin-Thorpe inequality.

Corollary 1.9. *Let (M, g) be a four-dimensional compact connected shrinking Ricci soliton satisfying (1.2). If*

$$\sqrt{\frac{R_{\max} - R_{\min}}{\lambda^2}}(16 + 6\pi^2) \leq \text{diam}(M, g), \quad (1.10)$$

then the soliton satisfies the Hitchin-Thorpe inequality $2\chi(M) \geq 3|\tau(M)|$.

This note is organized as follows: In Section 2, after introducing our notation, we shall prove Theorem 1.5 and Corollary 1.7. Ending with Section 3, a proof of Corollary 1.9 and some related result will be given.

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2. PRELIMINARIES

In this section, after introducing our notation, we shall prove Theorem 1.5 and Corollary 1.7. Let $X, Y, Z \in \mathfrak{X}(M)$ be three vector fields on M . For any smooth function $f \in C^\infty(M)$, the gradient vector field and Hessian of f are defined by

$$g(\nabla f, X) = df(X) \quad \text{and} \quad H_f(X, Y) = g(\nabla_X \nabla f, Y),$$

respectively. The curvature tensor and Ricci tensor are defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z \quad \text{and} \quad \text{Ric}(X, Y) = \sum_{i=1}^n g(R(e_i, X)Y, e_i),$$

respectively. Here, $\{e_i\}_{i=1}^n$ is an orthonormal frame of (M, g) . In order to prove Theorem 1.5, we will use the index form of a minimizing unit speed geodesic segment. We refer the reader to books [11, 14] for basic facts about this topic. The following Myers type theorem plays an important role in proving Theorem 1.5.

Theorem 2.1. *Let (M, g) be an n -dimensional complete connected Riemannian manifold. Suppose that (M, g) admits a smooth vector field V satisfying*

$$\text{Ric}_g + \mathcal{L}_V g \geq (n - 1)Cg \quad \text{and} \quad |V| \leq \gamma \quad (2.2)$$

for some constants $C > 0$ and $\gamma \geq 0$. Then (M, g) is compact and the diameter of (M, g) has the upper bound

$$\text{diam}(M, g) \leq \frac{4\gamma + \sqrt{16\gamma^2 + (n - 1)^2 C \pi^2}}{(n - 1)C}. \quad (2.3)$$

Remark 2.4. Under the same condition as in Theorem 2.1, Limoncu [12] gave the following diameter estimate

$$\text{diam}(M, g) \leq \frac{\pi}{(n-1)C} \left(\sqrt{2}\gamma + \sqrt{2\gamma^2 + (n-1)^2 C} \right). \quad (2.5)$$

Since

$$\sqrt{2}\pi \approx 4.44288 > 4 \quad \text{and} \quad 2\pi^2 \approx 19.73920 > 16,$$

our diameter estimate (2.3) is sharper than (2.5).

Proof of Theorem 2.1. Our proof of Theorem 2.1 is similar to that by Limoncu [12]. Take arbitrary two points $p, q \in M$. By the compactness of the manifold (M, g) , there exists the minimizing unit speed geodesic segment σ from p to q of length ℓ . Let $\{e_1 = \dot{\sigma}, e_2, \dots, e_n\}$ be a parallel orthonormal frame along σ . Recall that, for any smooth function $\phi \in \mathcal{C}^\infty([0, \ell])$ satisfying $\phi(0) = \phi(\ell) = 0$, we obtain

$$I(\phi e_i, \phi e_i) = \int_0^\ell \left(g(\dot{\phi} e_i, \dot{\phi} e_i) - g(R(\phi e_i, \dot{\sigma}) \dot{\sigma}, \phi e_i) \right) dt, \quad (2.6)$$

where $I(\cdot, \cdot)$ denotes the index form of σ . From (2.6), we have

$$\sum_{i=2}^n I(\phi e_i, \phi e_i) = \int_0^\ell \left((n-1) \dot{\phi}^2 - \phi^2 \text{Ric}_g(\dot{\sigma}, \dot{\sigma}) \right) dt, \quad (2.7)$$

where we have used $g(R(\dot{\sigma}, \dot{\sigma}) \dot{\sigma}, \dot{\sigma}) = 0$. By using the assumption (2.2) in the integral expression (2.7), we obtain

$$\begin{aligned} \sum_{i=2}^n I(\phi e_i, \phi e_i) &\leq \int_0^\ell \left((n-1)(\dot{\phi}^2 - C\phi^2) + \phi^2 (\mathcal{L}_V g)(\dot{\sigma}, \dot{\sigma}) \right) dt \\ &= \int_0^\ell \left((n-1)(\dot{\phi}^2 - C\phi^2) + 2\phi^2 g(\nabla_{\dot{\sigma}} V, \dot{\sigma}) \right) dt \\ &= \int_0^\ell \left((n-1)(\dot{\phi}^2 - C\phi^2) + 2\phi^2 \dot{\sigma}(g(V, \dot{\sigma})) \right) dt, \end{aligned} \quad (2.8)$$

where, the last equality follows from the parallelism of the metric g and $\nabla_{\dot{\sigma}} \dot{\sigma} = 0$. On the geodesic segment $\sigma(t)$, we have

$$\begin{aligned} 2\phi^2 \dot{\sigma}(g(V, \dot{\sigma})) &= 2\phi^2 \frac{d}{dt}(g(V, \dot{\sigma})) \\ &= -4\phi \dot{\phi} g(V, \dot{\sigma}) + 2 \frac{d}{dt}(\phi^2 g(V, \dot{\sigma})). \end{aligned} \quad (2.9)$$

Hence, by integrating both sides of (2.9), we have

$$\begin{aligned} \int_0^\ell 2\phi^2 \dot{\sigma}(g(V, \dot{\sigma})) dt &= \int_0^\ell -4\phi \dot{\phi} g(V, \dot{\sigma}) dt + [2\phi^2 g(V, \dot{\sigma})]_0^\ell \\ &= \int_0^\ell -4\phi \dot{\phi} g(V, \dot{\sigma}) dt \end{aligned} \quad (2.10)$$

$$\leq 4 \int_0^\ell \left| \phi \dot{\phi} g(V, \dot{\sigma}) \right| dt, \quad (2.11)$$

where, the second equality follows from $\phi(0) = \phi(\ell) = 0$. Since σ is a unit speed geodesic segment, the Cauchy-Schwarz inequality implies $|g(V, \dot{\sigma})| \leq |V|$. By combining this inequality and the assumption $|V| \leq \gamma$ in Theorem 2.1, we have $|g(V, \dot{\sigma})| \leq \gamma$. Hence, from (2.11) we obtain

$$\int_0^\ell 2\phi^2 \dot{\sigma}(g(V, \dot{\sigma})) dt \leq 4\gamma \int_0^\ell |\phi \dot{\phi}| dt. \quad (2.12)$$

From (2.8) and (2.12), we have

$$\sum_{i=2}^n I(\phi e_i, \phi e_i) \leq \int_0^\ell (n-1)(\dot{\phi}^2 - C\phi^2) dt + 4\gamma \int_0^\ell |\phi \dot{\phi}| dt. \quad (2.13)$$

If the function ϕ is taken to be $\phi(t) = \sin(\frac{\pi t}{\ell})$, then we obtain $\dot{\phi}(t) = \frac{\pi}{\ell} \cos(\frac{\pi t}{\ell})$ and

$$\phi \dot{\phi} = \frac{\pi}{\ell} \sin\left(\frac{\pi t}{\ell}\right) \cos\left(\frac{\pi t}{\ell}\right) = \frac{\pi}{2\ell} \sin\left(\frac{2\pi t}{\ell}\right).$$

Then, (2.13) becomes

$$\begin{aligned} \sum_{i=2}^n I(\phi e_i, \phi e_i) &\leq (n-1) \int_0^\ell \left(\frac{\pi^2}{\ell^2} \cos^2\left(\frac{\pi t}{\ell}\right) - C \sin^2\left(\frac{\pi t}{\ell}\right) \right) dt \\ &\quad + \frac{2\gamma\pi}{\ell} \int_0^\ell \left| \sin \frac{2\pi t}{\ell} \right| dt, \end{aligned}$$

and consequently, we have

$$\sum_{i=2}^n I(\phi e_i, \phi e_i) \leq -\frac{1}{2\ell} ((n-1)C\ell^2 - 8\gamma\ell - (n-1)\pi^2).$$

Since σ is a minimizing geodesic, we must obtain

$$(n-1)C\ell^2 - 8\gamma\ell - (n-1)\pi^2 \leq 0,$$

from where, we have

$$\ell \leq \frac{4\gamma + \sqrt{16\gamma^2 + (n-1)^2 C \pi^2}}{(n-1)C}.$$

This proves Theorem 2.1. □

Remark 2.14. Using Cauchy-Schwarz inequality, Limoncu estimated (2.10) from above by

$$\int_0^\ell 2\phi^2 \dot{\sigma}(g(V, \dot{\sigma})) dt = \int_0^\ell -4\phi \dot{\phi} g(V, \dot{\sigma}) dt \leq 4\sqrt{\int_0^\ell (\phi \dot{\phi})^2 dt} \sqrt{\int_0^\ell (g(V, \dot{\sigma}))^2 dt},$$

while we estimated (2.10) from above by an absolute value in (2.11) and obtained a better estimate (2.3) than (2.5).

The following lemma is useful to prove Theorem 1.5.

Lemma 2.15 (Fernández-López and García-Río [6]). *Let (M, g) be an n -dimensional compact shrinking Ricci soliton satisfying (1.2). Then*

$$|\nabla f|^2 \leq R_{\max} - R, \quad (2.16)$$

where R denotes the scalar curvature on the soliton.

Proof. We recall the proof for the reader's convenience. It is well-known that the potential function f of any gradient Ricci soliton (M, g) satisfies

$$R + |\nabla f|^2 - 2\lambda f = C \quad (2.17)$$

for some constant C , where R denotes the scalar curvature on the soliton. By compactness of the manifold M , there exists some global maximum point $p \in M$ of the potential function. Then, it follows from (2.17) that, for any point $x \in M$,

$$2\lambda f(p) = R(p) - C \geq 2\lambda f(x) = R(x) + |\nabla f|^2(x) - C, \quad (2.18)$$

and hence, $R(p) \geq R(x)$. Therefore, the scalar curvature also attains its maximum at p , and we obtain (2.16). \square

Now, we are in a position to prove Theorem 1.5.

Proof of Theorem 1.5. We apply Theorem 2.1 to the case that (M, g) is a compact gradient shrinking Ricci soliton. From (2.16), we have $|\nabla f| \leq \sqrt{R_{\max} - R_{\min}}$. Hence, by applying

$$V = \frac{1}{2}\nabla f, \quad C = \frac{\lambda}{n-1} \quad \text{and} \quad \gamma = \frac{1}{2}\sqrt{R_{\max} - R_{\min}}$$

to (2.3), we obtain (1.6). \square

Proof of Corollary 1.7. We show that

$$2\lambda f_{\max} - 2\lambda f_{\min} = R_{\max} - R_{\min}. \quad (2.19)$$

Although this equality was already proved by Fernández-López and García-Río in [5], we here show it for the reader's convenience. By using (2.17) and (1.2), we obtain

$$\text{Ric}_g(\nabla f, \cdot) = \frac{1}{2}dR. \quad (2.20)$$

By compactness of the manifold M , there exists some global minimum point $q \in M$ of the scalar curvature. From (2.20), we have $0 = (\nabla R)(q) = 2\text{Ric}_g(\nabla f, \cdot)(q)$. Since (M, g) has positive Ricci curvature, we have $(\nabla f)(q) = 0$. Then, it follows from (2.17) that, for any $x \in M$,

$$\begin{aligned} R(q) &= 2\lambda f(q) - |\nabla f|^2(q) + C = 2\lambda f(q) + C \\ &\leq R(x) = 2\lambda f(x) - |\nabla f|^2(x) + C \leq 2\lambda f(x) + C, \end{aligned}$$

from where we see that $q \in M$ is also a global minimum of the potential function, and hence, $R_{\min} = 2\lambda f_{\min} + C$. On the other hand, we have shown in (2.18) that $R_{\max} = 2\lambda f_{\max} + C$. Therefore, we obtain (2.19). Corollary 1.7 follows immediately from Theorem 1.5 and (2.19). \square

3. APPLICATIONS TO THEOREM 1.5

In this section, by using Theorem 1.5, we shall give a proof of Corollary 1.9. Throughout this section, we assume that (M, g) is a compact connected shrinking Ricci soliton satisfying (1.2). We use the following theorem to prove Corollary 1.9.

Theorem 3.1 (Ma [13]). *Let (M, g) be a four-dimensional compact shrinking Ricci soliton satisfying (1.2). If the scalar curvature satisfies*

$$\int_M R^2 \leq 24\lambda^2 \text{vol}(M, g),$$

then the soliton (M, g) satisfies the Hitchin-Thorpe inequality $2\chi(M) \geq 3|\tau(M)|$.

Proof of Corollary 1.9 . By taking the trace of (1.2), we have

$$R + \Delta f = 4\lambda. \quad (3.2)$$

Thanks to Theorem 1.5, the diameter of (M, g) has the upper bound

$$\text{diam}(M, g) < \frac{2}{\lambda} \sqrt{4(R_{\max} - R_{\min}) + 3\lambda\pi^2}. \quad (3.3)$$

Suppose that the inequality (1.10) holds. Then, from (3.3), we obtain

$$\frac{R_{\max} - R_{\min}}{\lambda^2} (16 + 6\pi^2) \leq \text{diam}^2(M, g) < \frac{4}{\lambda^2} \{4(R_{\max} - R_{\min}) + 3\lambda\pi^2\},$$

from where we have $R_{\max} < 6\lambda$. Hence, by (3.2), we have

$$\int_M R^2 \leq R_{\max} \int_M R < 24\lambda^2 \text{vol}(M, g),$$

and the result follows from Theorem 3.1. \square

The following result follows immediately from Theorem 1.5 and Theorem 3.1.

Corollary 3.4. *Let (M, g) be a four-dimensional compact connected shrinking Ricci soliton satisfying (1.2). If*

$$\frac{R_{\max}}{6\lambda} \cdot \frac{1}{\lambda} \left(2\sqrt{R_{\max} - R_{\min}} + \sqrt{4(R_{\max} - R_{\min}) + 3\lambda\pi^2} \right) \leq \text{diam}(M, g),$$

then the soliton satisfies the Hitchin-Thorpe inequality $2\chi(M) \geq 3|\tau(M)|$.

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